CLASSES OF WEIGHTED CONDITIONAL TYPE OPERATORS

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Abstract. In this paper, some classes of weighted conditional expectation type operators on $L^2(\Sigma)$ are characterized.

1. **Introduction and Preliminaries**

Let (X,Σ,μ) be a complete σ-finite measure space. For any sub-σ-finite algebra $\mathcal{A}\subseteq\Sigma$, the L^2-space $L^2(X,\mathcal{A},\mu_{|\mathcal{A}})$ is abbreviated by $L^2(\mathcal{A})$, and its norm is denoted by $\|.\|_2$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. The support of a measurable function f is defined as $S(f)=\{x\in X; f(x)\neq 0\}$. We denote the vector space of all equivalence classes of almost everywhere finite valued measurable functions on X by $L^0(\Sigma)$.

For a sub-σ-finite algebra $\mathcal{A}\subseteq\Sigma$, the conditional expectation operator associated with $\mathcal{A}$ is the mapping $f\to E^{\mathcal{A}}f$, defined for all non-negative measurable function f as well as for all $f\in L^2(\Sigma)$, where $E^{\mathcal{A}}f$, by the Radon-Nikodym theorem, is the unique $\mathcal{A}$-measurable function satisfying

$$\int_A f d\mu=\int_A E^{\mathcal{A}}f d\mu,\quad \forall A\in\mathcal{A}.$$

As an operator on $L^2(\Sigma)$, $E^{\mathcal{A}}$ is idempotent and $E^{\mathcal{A}}(L^2(\Sigma))=L^2(\mathcal{A})$. This operator will play a major role in our work. Let $f\in L^0(\Sigma)$, then f is said to be conditionable with respect to E if $f\in\mathcal{D}(E):=\{g\in L^0(\Sigma): E(|g|)\in L^0(\mathcal{A})\}$.

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Throughout this paper we take u and w in $\mathcal{D}(E)$. If there is no possibility of confusion, we write $E(f)$ in place of $E^{\mathcal{A}}(f)$. A detailed discussion about this operator may be found in [15].

Composition of conditional expectation operators and multiplication operators appear often in the study of other operators such as multiplication operators and weighted composition operators. Specifically, in [14], S.-T. C. Moy characterized all operators on L^p of the form $f \to E(fg)$ for g in L^q with $E(|g|)$ bounded. Eleven years later, R. G. Douglas, [2], analyzed positive projections on L^1 and many of his characterizations are in terms of combinations of multiplications and conditional expectations. More recently, P.G. Dodds, C.B. Huijsmans and B. De Pagter, [1], extended these characterizations to the setting of function ideals and vector lattices. J. Herron presented some assertions about the operator EM_u on L^p spaces in [10]. Also, some results about multiplication conditional expectation operators can be found in [9, 11]. In [3, 4] we investigated some classic properties of multiplication conditional expectation operators M_wEM_u on L^p spaces.

Let $\mathcal{H}$ be the infinite dimensional complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is a partial isometry if $\|Th\| = \|h\|$ for h orthogonal to the kernel of T. It is known that an operator T on a Hilbert space is partial isometry if and only if $TT^*T = T$. Every operator T on a Hilbert space $\mathcal{H}$ can be decomposed into $T = U|T|$ with a partial isometry U, where $|T| = (T^*T)^{\frac{1}{2}}$. U is determined uniquely by the kernel condition $\mathcal{N}(U) = \mathcal{N}(|T|)$. Then this decomposition is called the polar decomposition. The operator T is said to be positive operator and write $T \geq 0$, if $\langle Th, h\rangle \geq 0$, for all $h \in \mathcal{H}$.

In this paper we will be concerned with characterizing weighted conditional expectation type operators on $L^2(\Sigma)$ in terms of membership in the various partial normality classes. Here is a brief review of what constitutes membership for an

operator T on a Hilbert space in each classes:

(i) T belongs to $*$-A-class, if $|T^2| \geq |T^*|^2$.

(ii) T belongs to quasi-$*$-A-class, if $T^*|T^2|T \geq T^*|T^*|^2T$.

(iii) The operator T belongs to A-class, if $|T|^2 \leq |T^2|$.

There has been considerable interest in recent years in classes of operators that is mentioned above. A small sample of the related articles are found in our list of references ([5, 6, 8, 7, 13]).

2. Some classes of weighted conditional type operators

In the first we reminisce some theorems that we have proved in [4].

Theorem 2.1. The operator $T = M_wEM_u$ is bounded on $L^2(\Sigma)$ if and only if $(E|w|^2)^{\frac{1}{2}}(E|u|^2)^{\frac{1}{2}} \in L^\infty(\mathcal{A})$, and in this case its norm is given by $\|T\| = \|(E(|w|^2))^{\frac{1}{2}}(E(|u|^2))^{\frac{1}{2}}\|_\infty$.

Lemma 2.2. Let $T = M_wEM_u$ be a bounded operator on $L^2(\Sigma)$ and let $p \in (0, \infty)$. Then

$$(T^*T)^p = M_{\bar{u}(E(|u|^2))^{p-1}\chi_S(E(|w|^2))^p}EM_u$$

and

$$(TT^*)^p = M_{w(E(|w|^2))^{p-1}\chi_G(E(|u|^2))^p}EM_{\bar{w}},$$

where $S = S(E(|u|^2))$ and $G = S(E(|w|^2))$.

Theorem 2.3. The unique polar decomposition of bounded operator $T = M_wEM_u$ is $U|T|$, where

$$|T|(f)=\left(\frac{E(|w|^2)}{E(|u|^2)}\right)^{\frac{1}{2}}\chi_S\bar{u}E(uf)$$

and

$$U(f)=\left(\frac{\chi_{S\cap G}}{E(|w|^2)E(|u|^2)}\right)^{\frac{1}{2}}wE(uf),$$

for all $f\in L^2(\Sigma)$.

Theorem 2.4. The Aluthge transformation of $T=M_wEM_u$ is

$$\widehat{T}(f)=\frac{\chi_SE(uw)}{E(|u|^2)}\bar{u}E(uf), \qquad f\in L^2(\Sigma).$$

Remark 2.5. By Theorem 2.3 and Theorem 2.4, we can compute the polar decomposition and Aluthge transformation of $T^*=U^*|T^*|$ as follows:

$$|T^*|(f)=\left(\frac{E(|u|^2)}{E(|w|^2)}\right)^{\frac{1}{2}}\chi_G\bar{w}E(\bar{w}f);$$

$$U^*(f)=\left(\frac{\chi_{S\cap G}}{(E(|u|^2)E(|w|^2)}\right)^{\frac{1}{2}}\bar{u}E(\bar{w}f);$$

$$\widehat{T^*}(f)=\frac{\chi_GE(\overline{uw})}{E(|w|^2)}wE(\bar{w}f),$$

for all $f\in L^2(\Sigma)$.

In the sequel some necessary and sufficient conditions for weighted conditional type operators M_wEM_u to be in $*$-A-classes, quasi-$*$-A-classes and A-classes of operators, will be presented.

Theorem 2.6. Let $T=M_wEM_u$ be a bounded operator on $L^2(\Sigma)$. Then

(a) If $|E(uw)|^2\geq E(|u|^2)E(|w|^2)$ on $S=S(E(|u|^2))$, then T belongs to A-class.

(b) If T belongs to A-class, then $|E(uw)|^2 \geq E(|u|^2)E(|w|^2)$ on $S' = S(E(u))$.

(c) If $S = S'$, then T belongs to A-class if and only if $|E(uw)|^2 \geq E(|u|^2)E(|w|^2)$ on S.

Proof. (a) By Theorem 2.3 we have

$$|T|^2(f) = E(|w|^2)\chi_S\bar{u}E(uf), \qquad |T^2|(f) = |E(uw)|(\frac{E(|w|^2)}{E(|u|^2)})^{\frac{1}{2}}\chi_S\bar{u}E(uf),$$

for all $f \in L^2(\Sigma)$. So for every $f \in L^2(\Sigma)$ we have

$$\langle |T^2|(f) - |T|^2(f), f\rangle = \int_X |E(uw)|(\frac{E(|w|^2)}{E(|u|^2)})^{\frac{1}{2}}\chi_S\bar{u}\bar{f}E(uf) - E(|w|^2)\chi_S\bar{u}\bar{f}E(uf)d\mu$$

$$= \int_X |E(uw)|(\frac{E(|w|^2)}{E(|u|^2)})^{\frac{1}{2}}\chi_S|E(uf)|^2 - E(|w|^2)\chi_S|E(uf)|^2 d\mu.$$

This implies that if $|E(uw)|^2 \geq E(|u|^2)E(|w|^2)$ on S, then $\langle |T^2|(f) - |T|^2(f), f\rangle \geq 0$ for all $f \in L^2(\Sigma)$. So T is belonged to A-class.

(b) If T belongs to A-class, then by (a), for all $f \in L^2(\Sigma)$ we have

$$\int_X |E(uw)|(\frac{E(|w|^2)}{E(|u|^2)})^{\frac{1}{2}}\chi_S|E(uf)|^2 - E(|w|^2)\chi_S|E(uf)|^2 d\mu \geq 0.$$

Let $A \in \mathcal{A}$, with $0 < \mu(A) < \infty$. By replacing f to χ_A, we have

$$\int_A |E(uw)|(\frac{E(|w|^2)}{E(|u|^2)})^{\frac{1}{2}}\chi_S|E(u)|^2 - E(|w|^2)\chi_S|E(u)|^2 d\mu \geq 0.$$

Since $A \in \mathcal{A}$ is arbitrary, then $|E(uw)|^2 \geq E(|u|^2)E(|w|^2)$ on $S' = S(E(u))$.

(c) It follows from (a) and (b).$\square$

Theorem 2.7. Let $T = M_wEM_u$ be a bounded operator on $L^2(\Sigma)$. Then

(a) If

$$u|E(uw)|^{\frac{1}{2}}(\frac{E(|w|^2)}{E(|u|^2)})^{\frac{1}{4}}\chi_S \geq \bar{w}(E(|u|^2))^{\frac{1}{2}},$$

then T belongs to $*$-A-class.

(b) If T is belonged to $*$-A-class, then

$$|E(u)|^2|E(uw)|(\frac{E(|w|^2)}{E(|u|^2)})^{\frac{1}{2}}\chi_S \geq (E(|u|^2))^{\frac{1}{2}}|E(w)|^2.$$

Proof. (a) By using Theorem 2.3, for every $f \in L^2(\Sigma)$

$$|T^2|(f) = |E(uw)|(\frac{E(|w|^2)}{E(|u|^2)})^{\frac{1}{2}}\chi_S\bar{u}E(uf), \qquad |T^*|^2(f) = E(|u|^2)wE(\bar{w}f),$$

hence for every $f \in L^2(\Sigma)$

$$\langle |T^2|(f) - |T^*|^2(f), f\rangle = \int_X |E(uw)|(\frac{E(|w|^2)}{E(|u|^2)})^{\frac{1}{2}}\chi_S\bar{u}\bar{f}E(uf) - E(|u|^2)w\bar{f}E(\bar{w}f)d\mu$$

$$\int_X |E(uw)|(\frac{E(|w|^2)}{E(|u|^2)})^{\frac{1}{2}}\chi_S|E(uf)|^2 - E(|u|^2)|E(\bar{w}f)|^2d\mu.$$

This implies that, if

$$u|E(uw)|^{\frac{1}{2}}(\frac{E(|w|^2)}{E(|u|^2)})^{\frac{1}{4}}\chi_S \geq \bar{w}(E(|u|^2))^{\frac{1}{2}},$$

then $|T^2| \geq |T^*|^2$ i.e, T belongs to $*$-A-class.

(b) If T belongs to $*$-A-class, then by (a), for all $f \in L^2(\Sigma)$ we have

$$\int_X |E(uw)|(\frac{E(|w|^2)}{E(|u|^2)})^{\frac{1}{2}}\chi_S|E(uf)|^2 - E(|u|^2)|E(\bar{w}f)|^2d\mu \geq 0.$$

Let $A \in \mathcal{A}$, with $0 < \mu(A) < \infty$. By replacing f to χ_A, we have

$$\int_A |E(uw)|(\frac{E(|w|^2)}{E(|u|^2)})^{\frac{1}{2}}\chi_S|E(uf)|^2 - E(|u|^2)|E(\bar{w}f)|^2d\mu \geq 0.$$

Since $A \in \mathcal{A}$ is arbitrary, then

$$|E(u)|^2|E(uw)|(\frac{E(|w|^2)}{E(|u|^2)})^{\frac{1}{2}}\chi_S \geq (E(|u|^2))^{\frac{1}{2}}|E(w)|^2.$$

Theorem 2.8. Let $T = M_wEM_u$ be a bounded operator on $L^2(\Sigma)$. Then

(a) If $|E(uw)|^2 \geq E(|u|^2)E(|w|^2)$, then T belongs to quasi-$*$-A-class.

(b) If T belongs to quasi-$*$-A-class, then

$$|E(uw)|^3(E(|w|^2))^{\frac{1}{2}} \geq (E(|u|^2))^{\frac{3}{2}}(E(|w|^2))^2.$$

Proof. (a) Direct computation shows that, for $f \in L^2(\Sigma)$,

$$T^*|T^2|T(f) = |E(uw)|^3(\frac{E(|w|^2)}{E(|u|^2)})^{\frac{1}{2}}\chi_S\bar{u}E(uf), \quad T^*|T^*|^2T(f) = E(|u|^2)(E(|w|^2))^2\bar{u}E(uf).$$

So, for all $f \in L^2(\Sigma)$

$$\begin{aligned} &\langle T^*|T^2|T(f) - T^*|T^*|^2T(f), f\rangle \\ &= \int_X |E(uw)|^3(\frac{E(|w|^2)}{E(|u|^2)})^{\frac{1}{2}}\chi_S\bar{u}\bar{f}E(uf) - E(|u|^2)(E(|w|^2))^2\bar{u}\bar{f}E(uf)d\mu \\ &= \int_X (|E(uw)|^3(\frac{E(|w|^2)}{E(|u|^2)})^{\frac{1}{2}}\chi_S - E(|u|^2)(E(|w|^2))^2)|E(uf)|^2 d\mu. \end{aligned}$$

This implies that if $|E(uw)|^2 \geq E(|u|^2)E(|w|^2)$, then

$$T^*|T^2|T - T^*|T^*|^2T \geq 0.$$

(b) If T is belonged to quasi-$*$-A-class, then by (a), for all $f \in L^2(\Sigma)$ we have

$$\int_X |E(uw)|(\frac{E(|w|^2)}{E(|u|^2)})^{\frac{1}{2}}\chi_S|E(uf)|^2 - E(|u|^2)|E(\bar{w}f)|^2 d\mu \geq 0.$$

Let $A \in \mathcal{A}$, with $0 < \mu(A) < \infty$. By replacing f to χ_A, we have

$$\int_A |E(uw)|(\frac{E(|w|^2)}{E(|u|^2)})^{\frac{1}{2}}\chi_S|E(u)|^2 - E(|u|^2)|E(\bar{w})|^2 d\mu \geq 0.$$

Since $A \in \mathcal{A}$ is arbitrary, then

$$|E(uw)|^3(E(|w|^2))^{\frac{1}{2}} \geq (E(|u|^2))^{\frac{3}{2}}(E(|w|^2))^2.$$

J. Herron showed that: EM_u is normal if and only if $u \in L^\infty(\mathcal{A})$ [10]. So we have the following result.

Corollary 2.9. If we consider the operator $T = EM_u : L^2(\Sigma) \to L^2(\Sigma)$, then the followings are equivalent.

(a) T is normal.

(b) T is belonged to quasi-$*$-A-class.

(c) $u \in L^\infty(\mathcal{A})$.

Example 2.10. **(a)** Let $X = [0,1]\times[0,1]$, $d\mu = dxdy$, Σ the Lebesgue subsets of X and let $\mathcal{A} = \{A\times[0,1] : A \text{ is a Lebesgue set in } [0,1]\}$. Then, for each f in $L^2(\Sigma)$, $(Ef)(x,y) = \int_0^1 f(x,t)dt$, which is independent of the second coordinate. This example is due to A. Lambert and B. Weinstock [12]. Now, if we take $u(x,y) = y^{\frac{x}{8}}$ and $w(x,y) = \sqrt{(4+x)y}$, then $E(|u|^2)(x,y) = \frac{4}{4+x}$ and $E(|w|^2)(x,y) = \frac{4+x}{2}$. So,

$E(|u|^2)(x,y)E(|w|^2)(x,y) = 2$ and $|E(uw)|^2(x,y) = 64\frac{4+x}{(x+12)^2}$. Direct computations shows that $E(|u|^2)(x,y)E(|w|^2)(x,y) \leq |E(uw)|^2(x,y)$. Thus, by Theorem 2.6 the weighted conditional type operator M_wEM_u belongs to A-classes of operators on $L^2(\Sigma)$.

(b) Let $\Omega = [-1,1]$, $d\mu = \frac{1}{2}dx$ and $\mathcal{A} =< \{(-a,a) : 0 \leq a \leq 1\} >$ (Sigma algebra generated by symmetric intervals). Then

$$E^{\mathcal{A}}(f)(x) = \frac{f(x)+f(-x)}{2}, \quad x \in \Omega,$$

where $E^{\mathcal{A}}(f)$ is defined. Let $u(x) = x^2 - 1$ and $w \equiv 1$, then by Theorem 2.8 the operator M_wEM_u belongs to quasi-$*$-A-classes of operators on $L^2(\Sigma)$.

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